1. Optical set-up

The optical set-up is schematically described in Fig. 1. A distorted plane wave is focussed by a lens or a mirror $L_1$, with focal length $f$, on its focal plane $F$. The curvature sensor consists of two image detectors. One detect the irradiance distribution in plane $P_1$ at a distance $l$ before $F$. The other detect the irradiance distribution in plane $P_2$ at the same distance $l$ after $F$. For the sake of symmetry, a second lens $L_2$ of focal length $f/2$ is used in plane $F$ to reimage $L_1$ at a distance $f$ beyond $F$. It is shown that, at the geometrical approximation, the difference between the irradiance distributions in planes $P_1$ and $P_2$ is a measure of the local wavefront curvature and of the wavefront radial tilt at the edge of the pupil.

\[ A_0(\bar{r}) = P(\bar{r})\Psi(\bar{r})\exp -i\frac{\bar{r}^2}{\lambda f} \]

The complex amplitude diffracted in plane $P_1$ is given by [1]

\[ A_1(\bar{r}) = P(\bar{r})\Psi(\bar{r})\exp -i\frac{\bar{r}^2}{\lambda f} \cdot \frac{1}{i\lambda(f-l)} \exp i\lambda\frac{\bar{r}^2}{\lambda(f-l)} \]
where the convolution operator $*$ describes Fresnel diffraction over a distance $f-l$. Expressing the convolution as an integral gives

$$A_1(\tau) = \frac{1}{i\lambda(f-l)^2} \int P(\tau') \Psi(\tau') \exp\left(-i\pi \frac{\tau'^2}{\lambda f}\right) \exp\left(i\pi \frac{(\tau-f)^2}{\lambda(f-l)}\right) d\tau'$$

$$= \frac{1}{i\lambda(f-l)^2} \exp\left(i\pi \frac{\tau^2}{\lambda(f-l)}\right) \int P(\tau') \Psi(\tau') \exp\left(i\pi \frac{(\tau'-f)^2}{\lambda f}\right) d\tau'$$

$$\times \exp\left(2i\pi \frac{\tau' \tau}{\lambda(f-l)}\right) d\tau'$$

(3)

The related irradiance distribution is the square of the complex amplitude

$$I_1(\tau) = |A_1(\tau)|^2$$

$$= \frac{1}{\lambda^2(f-l)^2} \int \int P(\tau') P(\tau'') \Psi(\tau') \Psi^*(\tau'') \exp\left(2i\pi \frac{\tau' \tau''}{\lambda(f-l)}\right) d\tau' d\tau''$$

(4)

or, changing variable $\tau''$ into $\tau' + \bar{\rho}$, Eq. (4) becomes

$$I_1(\tau) = \frac{1}{\lambda^2(f-l)^2} \int \exp\left(-i\pi \frac{\tau'^2}{\lambda f}\right) \exp\left(2i\pi \frac{\bar{\rho} \tau'}{\lambda(f-l)}\right)$$

$$\times \int P(\tau') P(\tau' + \bar{\rho}) \Psi(\tau') \Psi^*(\tau' + \bar{\rho}) \exp\left(-2i\pi \frac{\bar{\rho} \tau'}{\lambda f}\right) d\tau' d\bar{\rho}$$

(5)

3. Irradiance distribution in plane $P_2$

The complex amplitude $A_F(\tau)$ diffracted in plane $F$ is obtained by setting $l$ equal to zero in Eq. (3)

$$A_F(\tau) = \frac{1}{i\lambda f} \exp\left(i\pi \frac{\tau^2}{\lambda f}\right) \int P(\tau') \Psi(\tau') \exp\left(-2i\pi \frac{\tau' \tau}{\lambda f}\right) d\tau'$$

(6)

or, after transmission through lens $L_2$,

$$A_F'(\tau) = \frac{1}{i\lambda f} \exp\left(-i\pi \frac{\tau^2}{\lambda f}\right) \int P(\tau') \Psi(\tau') \exp\left(-2i\pi \frac{\tau' \tau}{\lambda f}\right) d\tau'$$

(7)

The complex amplitude $A_2(\tau)$ diffracted further over a distance $l$ is obtained by convolving again with the Fresnel diffraction operator

$$A_2(\tau) = A_F'(\tau) * \frac{1}{i\lambda f} \exp\left(i\pi \frac{\tau^2}{\lambda f}\right)$$

(8)

which gives after a few manipulations

$$A_2(\tau) = \frac{1}{i\lambda(f-l)} \exp\left(-i\pi \frac{\tau^2}{\lambda(f-l)}\right) \int P(\tau') \Psi(\tau') \exp\left(-2i\pi \frac{\tau' \tau}{\lambda f}\right) d\tau'$$

$$\times \exp\left(-2i\pi \frac{\bar{\rho} \tau'}{\lambda(f-l)}\right) d\tau'$$

(9)

a relation very similar to (3). The related irradiance distribution is very similar to (5)

$$I_2(\tau) = \frac{1}{\lambda^2(f-l)^2} \int \exp\left(i\pi \frac{\bar{\rho}^2}{\lambda f}\right) \exp\left(2i\pi \frac{\bar{\rho} \tau}{\lambda f}\right)$$

$$\times \int P(\tau') \Psi(\tau') \exp\left(-2i\pi \frac{\bar{\rho} \tau'}{\lambda f}\right) d\tau' d\tau''$$

(10)
\begin{equation}
X \int P(\tau')P(\tau'+\rho)\Psi(\tau')\Psi(\tau'+\rho) \exp \frac{2i\pi}{\lambda f(f-l)} \frac{\rho \tau'}{\lambda f(f-l)} d\tau' d\rho
\end{equation}

3. The geometrical optics approximation

Let \( \rho_0 \) be the correlation length of the complex amplitude disturbance of the incoming wavefront. Physically, wavefront fluctuations of scale \( \rho_0 \) diffract light over an angle \( \lambda/\rho_0 \) and produce on plane \( P_1 \) a blur of size \( \lambda(f-l)/\rho_0 \). This blur must be small compared to the size of the fluctuations we want to measure which is \( \rho_0 \) scaled down by a factor \( l/f \), i.e. one must have

\begin{equation}
\frac{\lambda(f-l)}{\rho_0} \ll \frac{\rho_0 l}{f} \quad \text{or} \quad \frac{\lambda f(f-l)}{l \rho_0^2} \ll 1
\end{equation}

We assume that this condition is met in the following.

The contribution to the integral over \( \tau' \) in Eqs (5) or (10) takes significant values only when \( \lambda f(f-l)/l|\rho| \) is larger than or of the order of \( \rho_0 \), i.e. when

\begin{equation}|\rho| \leq \frac{\lambda f(f-l)}{\rho_0 l}
\end{equation}

When condition (11) is met this occurs when \( |\rho| \ll \rho_0 \), hence the product \( \Psi(\tau')\Psi(\tau'+\rho) \) in Eqs (5) and (10) can be replaced by an approximate expression valid only when \( |\rho| \ll \rho_0 \). Assuming pure phase disturbances and calling \( \phi(\tau') \) the phase of the incoming wavefront, we have

\begin{equation}
\Psi(\tau')\Psi'(\tau'+\rho) = \exp -i \left( \phi(\tau'+\rho) - \phi(\tau') \right)
\end{equation}

or, for \( |\rho| \ll \rho_0 \),

\begin{equation}
\Psi(\tau')\Psi'(\tau'+\rho) \approx \exp -i \rho \nabla \phi(\tau')
\end{equation}

\approx 1 - i \rho \nabla \phi(\tau')

Since \( \rho_0 \) is much smaller than the entrance pupil, under the same conditions, we also have

\begin{equation}
P(\tau')P(\tau'+\rho) \approx P(\tau')
\end{equation}

Hence, Eq. (5) becomes

\begin{equation}
I_1(\tau) = \frac{1}{\lambda^2(f-l)^2} \int \exp -i \pi \frac{\rho^2}{\lambda f(f-l)} \exp 2i\pi \frac{\rho \tau'}{\lambda f(f-l)} 
\times \int P(\tau') \left( 1 - i \rho \nabla \phi(\tau') \right) \exp -2i\pi \frac{\rho \tau'}{\lambda f(f-l)} d\tau' d\rho
\end{equation}

The second integral in Eq. (16) is given by the Fourier transform of \( P(\tau') \)

\begin{equation}
\left( 1 - i \rho \nabla \phi(\tau') \right)
\end{equation}

which becomes negligibly small at frequencies larger than \( 1/\rho_0 \) Hence this integral takes significant values only for

\begin{equation}
|\rho| \leq \frac{\lambda f(f-l)}{l \rho_0}
\end{equation}

in which case, according to (11), the quadratic phase in the first integral becomes very small compared to unity

\begin{equation}
\frac{\rho^2}{\lambda f(f-l)} \leq \frac{\lambda f(f-l)}{l \rho_0^2} \ll 1
\end{equation}
and can be neglected, yielding

$$I_1(\vec{r}) = \frac{1}{\lambda^2(f-l)^2} \int \exp 2i\pi \frac{\vec{p} \cdot \vec{r}}{\lambda(f-l)} \times \int P(\vec{r}') \left(1 - i\vec{p} \nabla \phi(\vec{r}')\right) \exp -2i\pi \frac{\vec{p} \cdot \vec{r}'}{\lambda f(f-l)} \, d\vec{r}' d\vec{p}$$

(18)

Similarly, Eq. (10) gives

$$I_2(\vec{r}) = \frac{1}{\lambda^2(f-l)^2} \int \exp 2i\pi \frac{\vec{p} \cdot \vec{r}}{\lambda(f-l)} \times \int P(\vec{r}') \left(1 - i\vec{p} \nabla \phi(\vec{r}')\right) \exp 2i\pi \frac{\vec{p} \cdot \vec{r}'}{\lambda f(f-l)} \, d\vec{r}' d\vec{p}$$

(19)

The approximation we have made is called the geometrical approximation since it yields results predicted by geometrical optics. It was used by Reiger in his early theory of stellar scintillation [2].

4. The sensor signal

When the incoming wavefront is a plane wave ($\phi=0$), Eqs (18) gives

$$I(\vec{r}) = \frac{1}{\lambda^2(f-l)^2} \int \exp 2i\pi \frac{\vec{p} \cdot \vec{r}}{\lambda(f-l)} P(\vec{r}') \exp 2i\pi \frac{\vec{p} \cdot \vec{r}'}{\lambda f(f-l)} \, d\vec{r}' d\vec{p}$$

$$= (f/l)^2 P(f\vec{r}/l)$$

(20)

i.e. the illumination is uniform over a reduced size pupil "image" with brightness $(f/l)^2$ times the entrance pupil brightness as expected from geometrical optics. Similarly Eq. (19) gives

$$I(\vec{r}) = (f/l)^2 P(-f\vec{r}/l)$$

(21)

describing an inverted but otherwise identical pupil image. When the incoming beam is distorted, the illumination $I_1(\vec{r})$ in plane $P_1$ given by Eq. (18) is the sum of the above uniform illumination plus a fluctuation

$$\Delta I_1(\vec{r}) = \frac{-i}{\lambda^2(f-l)^2} \int \exp 2i\pi \frac{\vec{p} \cdot \vec{r}}{\lambda(f-l)} \times \vec{p} \cdot \int P(\vec{r}') \nabla \phi(\vec{r}') \exp -2i\pi \frac{\vec{p} \cdot \vec{r}'}{\lambda f(f-l)} \, d\vec{r}' d\vec{p}$$

(22)

Changing $\vec{p}$ into

$$\vec{p} = \frac{\lambda f(f-l)}{l} \vec{u},$$

gives

$$\Delta I_1(\vec{r}) = \frac{-i \lambda f^3(f-l)}{l^3} \int \exp 2i\pi \frac{f}{l} \vec{u} \cdot \vec{r} \times \vec{u} \cdot \int P(\vec{r}') \nabla \phi(\vec{r}') \exp -2i\pi \vec{u} \cdot \vec{r}' \, d\vec{r}' d\vec{u}$$

(23)
\[
\Delta I_1(\vec{r}) = -\frac{\lambda f^3(f-l)}{2\pi l^3} \nabla \left[ P(f\vec{r}/l) \nabla \phi(f\vec{r}/l) \right] \\
= -\frac{\lambda f^3(f-l)}{2\pi l^3} \left[ \frac{\partial}{\partial \vec{r}} \phi(f\vec{r}/l) \delta_c + \nabla^2 \phi(f\vec{r}/l) \right] 
\]

(24)

where \(\delta_c\) represents a linear impulse distribution around the pupil edge. The derivative \(\partial \phi/\partial \vec{r}\) is the radial wavefront tilt at the pupil edge.

Similarly, Eq. (19) describes the illumination \(I_2(\vec{r})\) in plane \(P_2\) as a sum of a uniform illumination \(I(\vec{r})\) given by Eq. (21) plus a fluctuation

\[
\Delta I_2(\vec{r}) = \frac{\lambda f^3(f-l)}{2\pi l^3} \left[ \frac{\partial}{\partial \vec{r}} \phi(-f\vec{r}/l) \delta_c + \nabla^2 \phi(-f\vec{r}/l) \right] 
\]

(25)

We take as the sensor signal \(S(\vec{r})\) the difference \(I_2(\vec{r})-I_1(-\vec{r})\) between the illuminations in planes \(P_1\) and \(P_2\) normalized with \(I(\vec{r})\) or, using Eqs (20), (24) and (25)

\[
S(\vec{r}) = \frac{\Delta I_2(\vec{r}) - \Delta I_1(-\vec{r})}{I(\vec{r})} \\
= \frac{-\lambda f(f-l)}{4\pi l} \left[ \frac{\partial}{\partial \vec{r}} \phi(f\vec{r}/l) \delta_c + \nabla^2 \phi(f\vec{r}/l) \right] 
\]

(27)

Eq. (27) shows that the sensor will map the wavefront curvature \(\nabla^2 \phi\) on the pupil and the wavefront radial tilt \(\partial \phi/\partial \vec{r}\) at the edge. This information allows complete wavefront retrieval by solving the Poisson equation with the observed boundary conditions.

5. References