AO 2001: Theory vs. Experiment

Incoherency and Multiple Laser Guide Stars

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Laser Guide Stars

- Rayleigh and sodium laser guide stars are incoherent radiators

  — meaning —

- The backscattering or resonant re-radiation are incoherent processes.

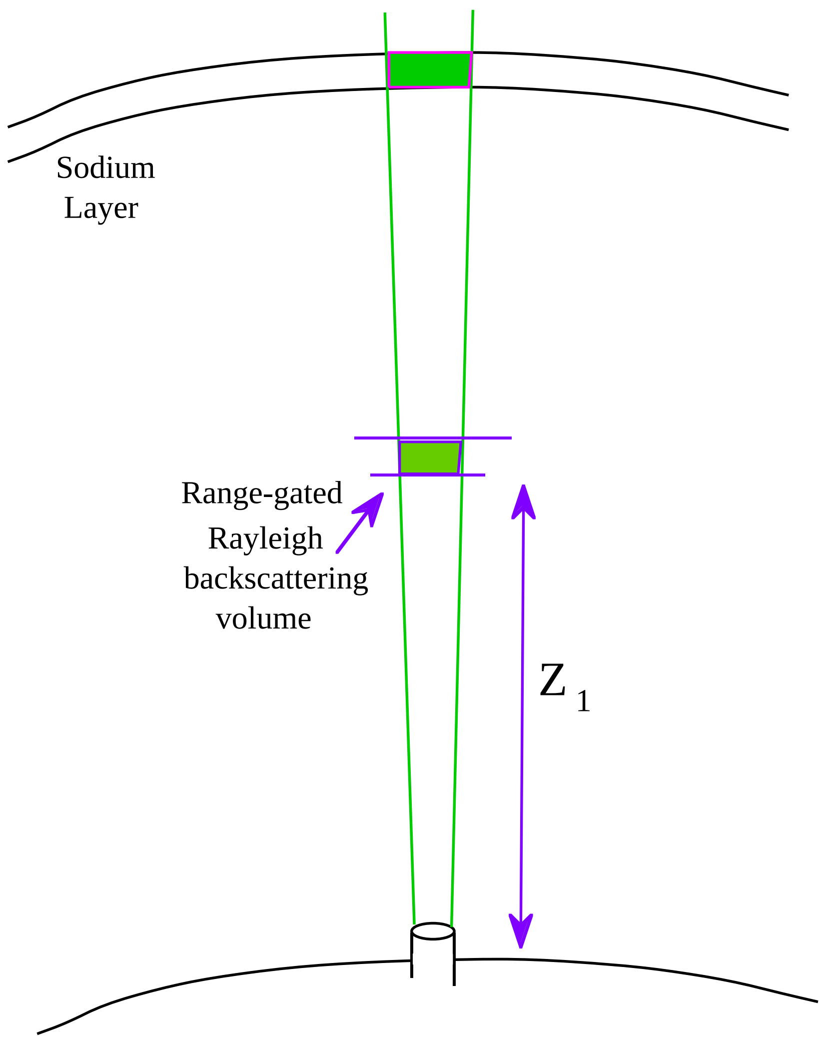

Figure 1

Rayleigh LGS

- The remaining coherence of the uplink LGS beam is destroyed upon "reflection" because of the irregular distribution of the scattering centers.

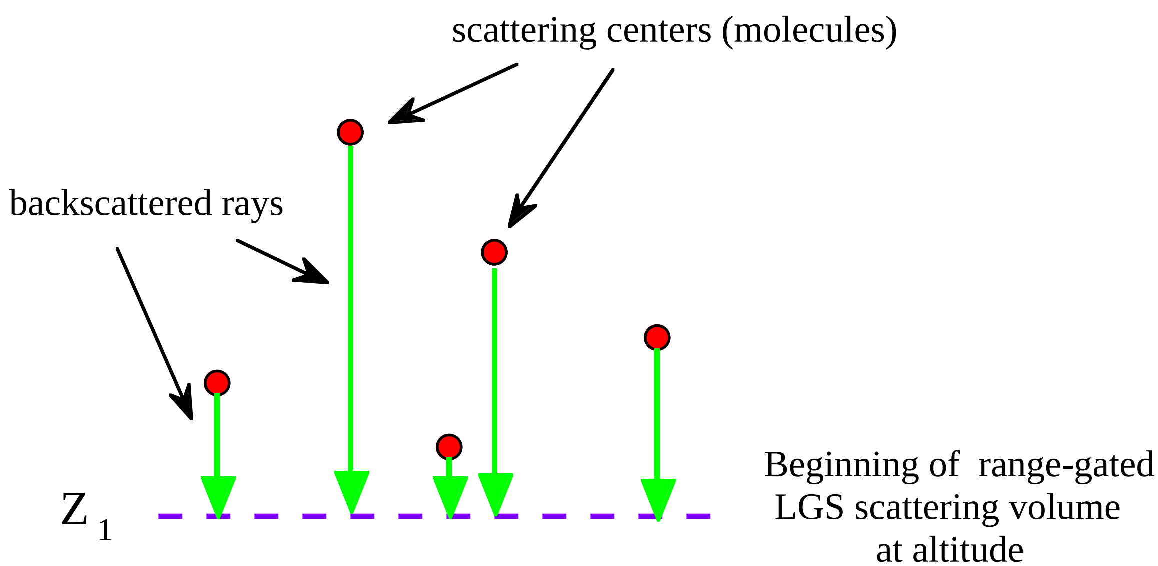

Figure 2

- The backscattering rays all have different optical path lengths back to the beginning of the scattering volume.
- Another indication that the scattered light is incoherent is that it is generally found that in a gas containing N molecules per unit volume, the total intensity scattered per unit volume is N times that scattered by a single molecule. Thus the total electric field amplitude is not the sum of the individual field amplitudes, but instead the sum of the square of the individual amplitudes. This implies that the phases have a random relationship.

Sodium LGS

- The resonant re-radiation in the sodium layer is also an incoherent process.

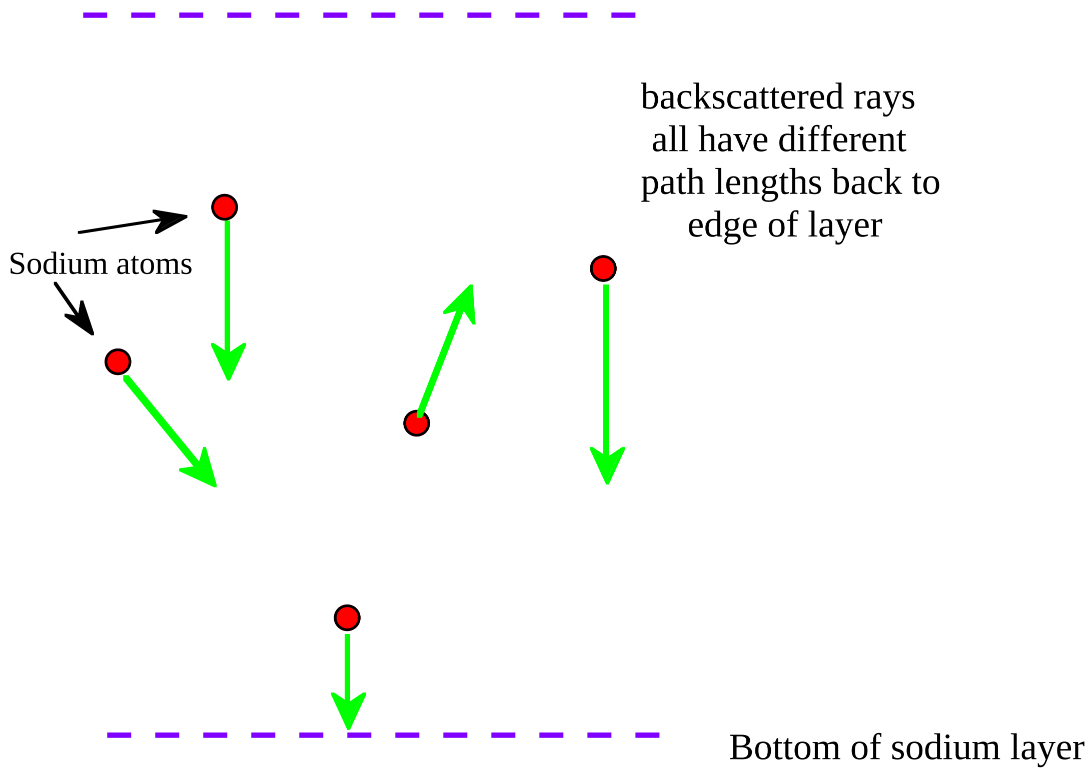

Figure 3

- Not only do the backscattered rays all have different optical path lengths back to the bottom edge of the sodium layer, but the lifetime of the excited state is 16 nanoseconds, hence the re-radiation is also scattered in time (light moves roughly 5 meters in this time.

LGS Return Initial Conditions

- For simplicity, we will assume that the return backscattered LGS wavefront, diameter D_{LGS}, exiting the bottom edge of the scattering layer is uniform in intensity and completely incoherent in phase.

- The LGS propagates back the receiver and AO system through atmospheric turbulence. The AO WFS measures the phase front of the LGS.

- **The Problem:** The AO system cannot distinguish the contribution of the atmospheric turbulence to the phase aberrations from that of the LGS initial conditions at altitude.

- As we shall see, in the ray-optics limit, the LGS system completely fails.

Ray-Optics Limit

- First, let's consider the uncompensated case. The light from a distant star propagates into the Earth's atmosphere to the bottom of the backscatter layer at $z = L$ (phase front red in Figure 4). For simplicity, let's assume that there is little turbulence above $z = L$ so that ϕ_0 is still essentially a plane wave.

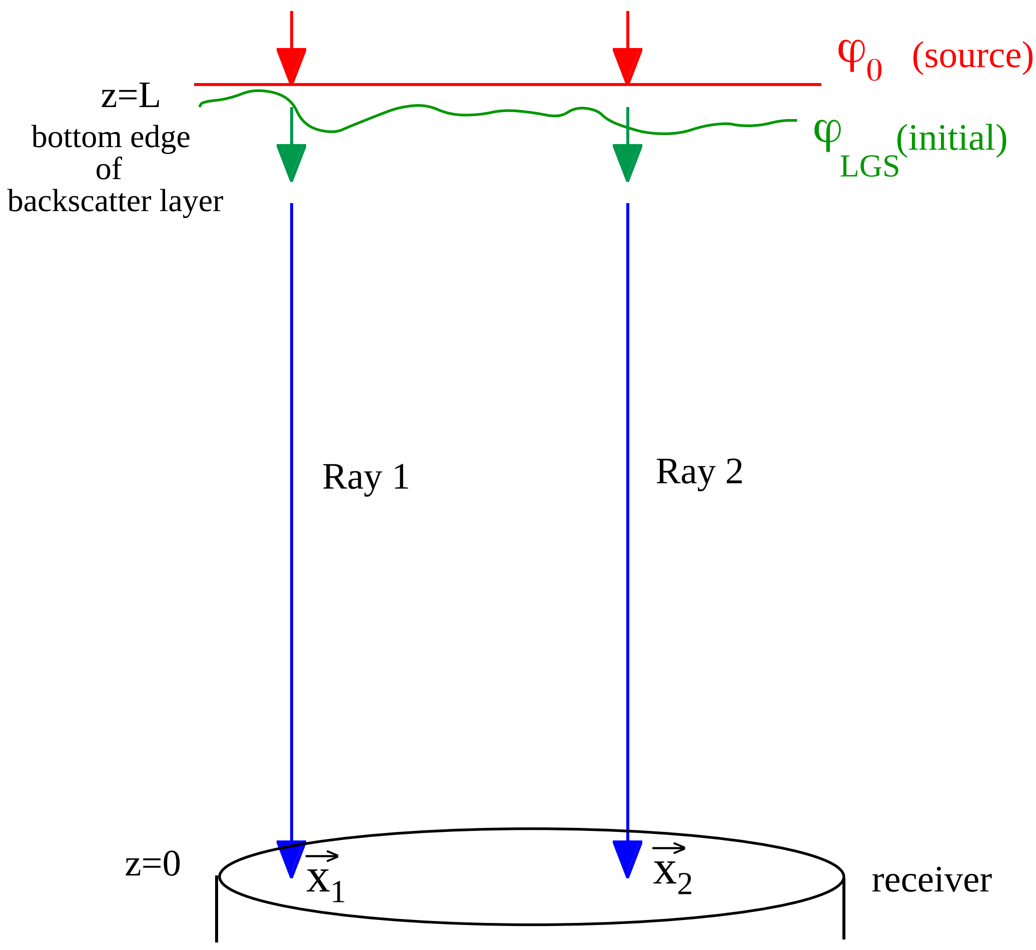

Figure 4

Ray-Optics Computation

- The light travels through the atmosphere to a telescope at $z = 0$. When it reaches the telescope, the phase of the star light along ray 1 is (ray-optics limit)

$$\phi_{\rm star}(\vec{x}_1, z = 0) = \phi_0 + k \int_L^0 \delta n(\vec{x}_1, z) dz, \tag{1}$$

  where δn is the fluctuation in the index of refraction in the atmosphere due to turbulence. Along ray 2 replace $\vec{x}_1$ with $\vec{x}_2$.

- In the ray-optics limit, the complex amplitude of the star light entering the telescope at $z = 0$ is

$$U_{\rm star}(\vec{x}_1, 0) \approx \exp(i\phi_{\rm star}(\vec{x}_1, 0)). \tag{2}$$

- Thus, the Mutual Coherence Function, Γ is

$$\begin{aligned} \Gamma(\vec{x}_1, \vec{x}_2) &= U^*_{\rm star}(\vec{x}_1, 0)\, U_{\rm star}(\vec{x}_2, 0) \\ &= \exp(i(\phi_{\rm star}(\vec{x}_2, 0) - \phi_{\rm star}(\vec{x}_1, 0)) \\ &= \exp(ik \int_L^0 dz(\delta n(\vec{x}_2, z) - \delta n(\vec{x}_1, z))) \end{aligned} \tag{3}$$

Ray-Optics Computation – 2

- Assuming that the index fluctuations are Gaussian, the ensemble averaged MCF is

$$\begin{aligned} \langle\Gamma(\vec{x}_1,\vec{x}_2)\rangle &= \exp(-\frac{k}{2}\int_L^0 dz\langle(\delta n(\vec{x}_2,z)-\delta n(\vec{x}_1,z))^2\rangle) \\ &= \exp(-(|\vec{x}_1-\vec{x}_2|/\rho_0)^{5/3}), \end{aligned} \tag{4}$$

  where ρ_0 is the usual coherence length, and where we assume the usual Kolmogorov structure function, $\langle(\delta n(r_1)-\delta n(r_2))^2\rangle = C_n^2\,|r_1-r_2|^{2/3}$.

- The expected Strehl ratio, $\langle S\rangle$ is

$$\langle S\rangle = \frac{16}{\pi^2 D^4}\int d^2x_1\; d^2x_2\langle\Gamma(\vec{x}_1,\vec{x}_2)\rangle, \tag{5}$$

  where D is the diameter of the telescope.

- In the limit of completely coherent light, $\rho_0 \rightarrow \infty$. In the limit of complete incoherence, $\rho_0 \rightarrow 0$. Atmospheric turbulence typically produces $\rho_0 = 2$–$5\ cm$ for visible light.

- For a coherent plane wave, $\langle S\rangle = 1$. The uncompensated Strehl ratio through typical atmospheric turbulence in the visible for a telescope of diameter 1.5 m is less than 0.005.

Ray-Optics Computation – 3

- Now let's add the compensation. First, we do the same ray-optics calculation for the LGS. When the LGS light reaches the telescope along ray 1, the phase is

$$\phi_{\mathrm{LGS}}(\vec{x}_1, z=0) = \phi_{\mathrm{LGS}}(\vec{x}_1, L) + k\int_L^0 \delta n(\vec{x}_1, z)dz. \tag{6}$$

  Along ray 2 replace $\vec{x}_1$ with $\vec{x}_2$.

- The phase of the LGS beacon light at $z = 0$ is conjugated and applied to the incoming star light. For *perfect* phase compensation, the compensated starlight at $z = 0$ is

$$\begin{aligned}\phi_{\mathrm{comp}}(\vec{x}_1, 0) &= \phi_{\mathrm{star}}(\vec{x}_1, 0) - \phi_{\mathrm{LGS}}(\vec{x}_1, 0) \\ &= \phi_0 + k\int_L^0 \delta n(\vec{x}_1, z)dz \\ &\qquad - \ \phi_{\mathrm{LGS}}(\vec{x}_1, L) - k\int_L^0 \delta n(\vec{x}_1, z)dz \\ &= \phi_0 - \ \phi_{\mathrm{LGS}}(\vec{x}_1, L)\end{aligned} \tag{7}$$

- Thus, the compensated MCF for the starlight is

$$\Gamma_{\mathrm{comp}}(\vec{x}_1, \vec{x}_2) = \exp(i(\phi_{\mathrm{LGS}}(\vec{x}_1, L) \ - \ \phi_{\mathrm{LGS}}(\vec{x}_2, L))). \tag{8}$$

Ray-Optics Conclusion

- Therefore, in the ray-optics limit, the "perfectly" compensated MCF depends *entirely* on the coherence of LGS at altitude $z = L$.

- However, the LGS is completely incoherent at $z = L$, i.e. the phase at two points $\vec{x}_1$ and $\vec{x}_2$ is completely uncorrelated.

- The compensated MCF is therefore a delta function. This is equivalent to taking the limit $\rho_0 \to 0$, hence the compensated Strehl $\langle S \rangle \to 0$.

- Compensation fails because the LGS is an incoherent radiator at $z = L$.

Diffraction to the Rescue

- First we note from equation (8) that if the LGS were coherent at altitude that the compensated Strehl ratio would be 1. (In the ray-optics limit phase compensation is all that is need to carry out propagation reciprocity.)

- Next we note that if the incoherency of the LGS at altitude was somehow mitigated by the time it reached the receiver (where it is actually measured), then the compensated Strehl would improve. There is no natural mechanism to do this in the ray-optics limit, however,

- Diffraction restores some coherency via propagation.

- From equation (8) we can see that we are interested in the propagation of the mutual coherence function of the LGS from altitude to ground.

- The van-Cittert–Zernike theorem deals with the propagation of the mutual coherence function and quantitatively expresses the effect of diffraction on the propagation of the LGS to the ground.

van Cittert – Zernike Theorem

- Before we state the result from the van Cittert–Zernike theorem that we need, let us first define the complex degree of spatial coherence, $\gamma(\vec{x}_1, \vec{x}_2) = \gamma(\vec{x}_1 - \vec{x}_2)$. The complex degree of spatial coherence is just the mutual coherence function Γ normalized by the intensity:

$$\gamma(\vec{x}_1, \vec{x}_2) = \frac{\langle U^*(\vec{x}_1, z) U(\vec{x}_2, z)\rangle}{\sqrt{I(\vec{x}_1, z) I(\vec{x}_2, z)}}, \tag{9}$$

  where $I(\vec{x}_1, z) = U^*(\vec{x}_1, z)U(\vec{x}_1, z)$ is the intensity at the point $(\vec{x}_1, z)$.

- Consider an incoherent source with intensity distribution $I(\vec{s}_1)$ on the $z = L$ plane. The van-Cittert–Zernike theorem states that the modulus of the complex degree of spatial coherence in the $z = 0$ plane is given by

$$|\gamma(|\vec{x}_1 - \vec{x}_2|)| = \frac{|\int d\vec{s}_1 \; I(\vec{s}_1) \exp(-ik(\vec{s}_1 \cdot (\vec{x}_1 - \vec{x}_2))/L)|}{\int d\vec{s}_1 I(\vec{s}_1)}, \tag{10}$$

  as long as the angular extent of the source viewed from the $z = 0$ plane is small.

- In other words, the van Cittert–Zernike theorem says that the modulus of the complex degree of spatial coherence for sources of small angular extent is equal to the modulus of the normalized Fourier transform of the intensity distribution of the source.

van Cittert – Zernike Theorem – 2

- For a concrete example, consider an circular incoherent source of diameter d with a uniform intensity distribution in the $z = L$ plane. In the $z = 0$ plane, the modulus of the complex degree of coherence is

$$|\gamma(r)| = \frac{2J_1(k\alpha r/2)}{k\alpha r/2}, \tag{11}$$

  where $\alpha = d/L$ is the angular "size" of the source as seen at a distance L, and $r = |\vec{x}_1 - \vec{x}_2|$.

- One can define a coherency length ρ_c as the distance within the wavefront over which the beam is now coherent after propagating a distance L from the source. From equation (11) we can take ρ_c to be the distance at which $|\gamma|$ first vanishes: $|\gamma(\rho_c)| = 0$. In this case,

$$\rho_c = 1.22\lambda/\alpha. \tag{12}$$

- From equation (12) we can see that the larger the angular size α, the smaller the degree of coherency achieved upon propagation. For a fixed distance L from a source, the larger the incoherent source, the smaller the degree of coherency achieved a distance L away.

Michelson Stellar Interferometer

- To a good approximation, stars are incoherent radiators with a uniform intensity over a circular disk.

- The Michelson stellar interferometer is a device to measure $|\gamma(r)|$ for light reaching us from distant stars and determine the angular diameter of the star from equation (12).

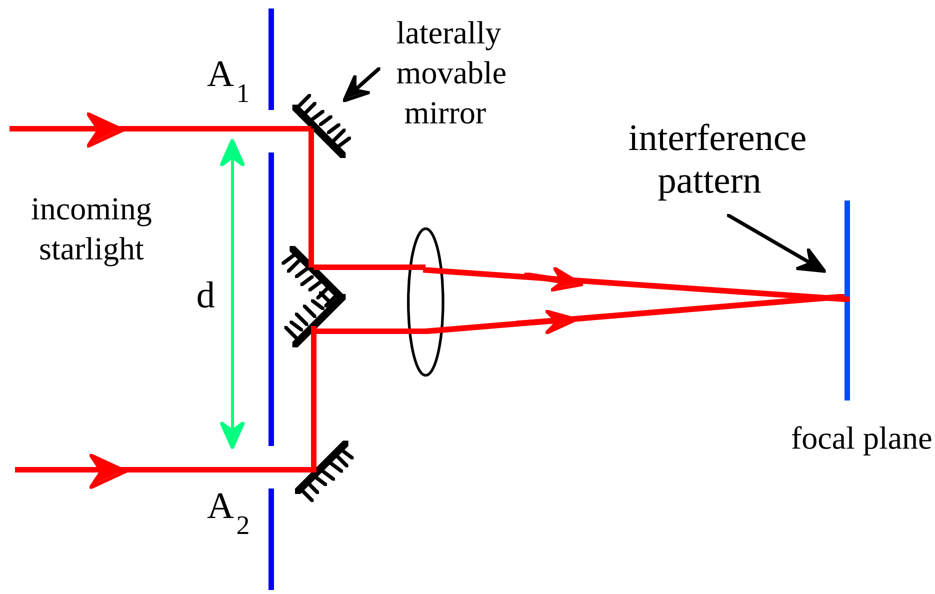

Figure 5

- Two points in the starlight wavefront are sampled at two small apertures A_1 and A_2 a distance d apart. The light from the two apertures is combined in the focal plane of an objective behind the plane of the apertures. The combined light forms an interference pattern. The visibility of the fringes is directly related to $|\gamma(r)|$. The separation d is increased until the fringes first disappear. When this happens, $d = \rho_c$.

Application to LGS

- We have assumed that the LGS at altitude is a disk of diameter D_{LGS} with uniform intensity. The angular extent of the LGS is $\alpha_{\mathrm{LGS}} = D_{\mathrm{LGS}}/L$.

- Let us still assume *perfect* compensation. To include the effect of diffraction in restoring partial coherency to the LGS as it propagates back, then we need to replace the normalized compensated MCF in equation (8),

$$\gamma_{\mathrm{comp}}(\vec{x}_1 - \vec{x}_2) = \frac{\Gamma_{\mathrm{comp}}(\vec{x}_1 - \vec{x}_2)}{I_{\mathrm{LGS}}}, \tag{13}$$

  with equation (11),

$$\gamma_{\mathrm{comp}}(\vec{x}_1 - \vec{x}_2) \rightarrow |\gamma(r)|. \tag{14}$$

- The compensated Strehl ratio in this case for a telescope of diameter D is

$$\begin{aligned} \langle S \rangle &= \frac{16}{\pi^2 D^4} \int_D d^2x_1 \int_D d^2x_2 \, \big|\gamma(|\vec{x}_1 - \vec{x}_2|)\big| \\ &= \frac{16}{\pi} \int_0^1 dq \, q \left(\cos^{-1}(q) - q\sqrt{1-q^2} \right) \big|\gamma(qD)\big|. \end{aligned} \tag{15}$$

Application to LGS – 2

- Let us display the results in terms of a normalized LGS diameter at altitude. For a telescope of diameter D, the diffraction limited spot size or diameter, $\mathcal{D}$, of the LGS at altitude L in the absence of turbulence is

$$\mathcal{D} = 2.44\,\frac{\lambda L}{D}. \tag{16}$$

- When $D_{\mathrm{LGS}} = \mathcal{D}$, then $\rho_c = D/2$, the radius of the telescope aperture. This is the best that diffraction will allow.

$D_{\mathrm{LGS}}/\mathcal{D}$	Strehl $\langle S \rangle$
1	0.59
1.5	0.33
2	0.21

- Recall that the results in the table above do not include the effects of turbulence – we are assuming perfect compensation (the phase is measured with infinite accuracy and applied with infinite accuracy, no amplitude effects, etc.) The Strehls above only include the effects of the incoherency of the beacon at altitude. This incoherency is due to the fact that the back scattering or resonant re-radiation is inherently an incoherent process.

Application to LGS – 3

- It is apparent that for a given altitude L of the backscatter layer, the larger the LGS diameter, the worse the effects of the incoherency.

- On the other hand, the smaller the LGS diameter, the worse the effects of focal anisoplanatism.

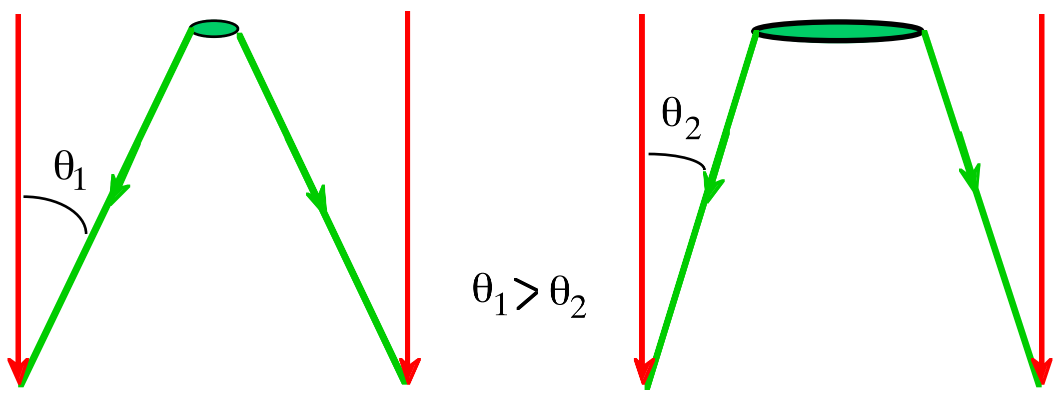

Figure 6

- A Rayleigh LGS has the backscatter layer at a lower altitude than the Sodium LGS layer. Therefore, diffraction has less distance to mitigate the effects of incoherency through propagation back to the telescope for a Rayleigh LGS than a Sodium LGS. In order to match the coherency restored for a Sodium LGS, the Rayleigh LGS diameter must be smaller, and the focal anisoplanatism correspondingly larger.

Multiple LGS

- The use of multiple LGS in a constellation to decrease focal anisoplanatism and increase the field of view is equivalent to increasing the incoherent LGS diameter.

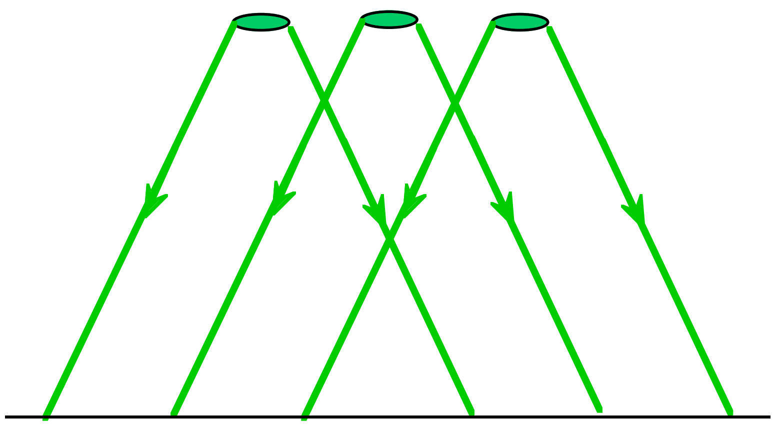

Figure 7

- Thus what is gained by having a larger beacon is lost to incoherency.

- In order to avoid the increase in the effects of incoherency, the various components of the LGS constellation must be distinguishable.

- For a Rayleigh LGS this could be accomplished by using slightly different colors for each LGS component. For sodium LGS, a sequence with slight temporal differences between components could be used. Both require increased computation to calculate the correction.

Further Consequences of Incoherency

- The same effect of diffraction that wipes out some of the LGS incoherency at altitude will also wipe out the effect of high altitude turbulence.
- Incoming plane waves from astronomical sources are equivalent to large beacons, except they are initially coherent. The difference in "size" between the incoming plane waves and the LGS means that the effects of diffraction on the high altitude turbulence induced aberrations will not be the same in the plane wave as in the LGS. Thus, the high altitude turbulence will not be properly corrected (even with full-field compensation).
- From SCIDAR results (http://op.ph.ic.ac.uk), there appears to be significant high altitude turbulence (for example, between 14 and 16 Km above SOR).

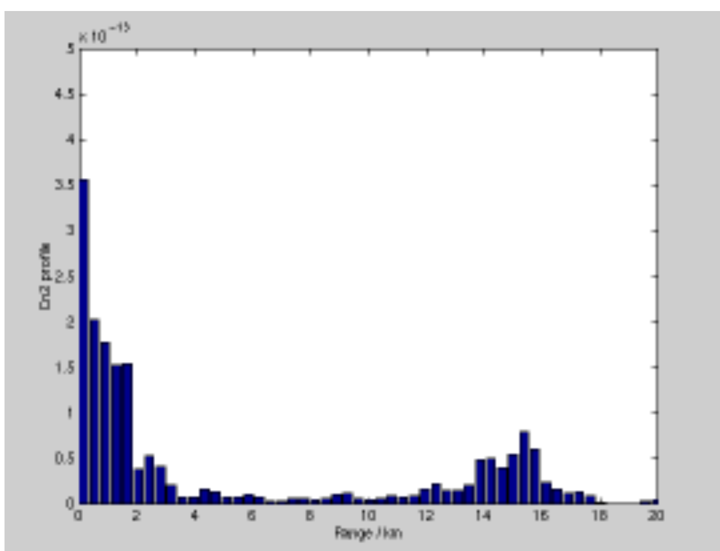

Figure 8 Profile above SOR. Reproduced from http://op.ph.ic.ac.uk

Further Consequences – 2

- However, the SCIDAR technique for measuring the C_n^2 turbulence profile appears to produce results that fluctuate rapidly with time.

- C_n^2 is defined using an *ensemble averaged* 2-point function, therefore it is a quantity that characterises the entire ensemble distribution and *not* individual ensemble members or realizations of turbulence. Thus, unless the ensemble distribution is changing, C_n^2 in a local neighborhood or a layer is a constant.

- Thus the SCIDAR data indicates that the measurements represent ensemble members, and not technically C_n^2.

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Incoherency and Multiple Laser Guide Stars

Outline

- Rayleigh and sodium laser guide stars are incoherent radiators.
- Incoherency of LGS is indistinguishable from effects of atmospheric turbulence, yet we do not want to correct astronomical sources with the initial conditions of the LGS mixed in.
- Ray-optics calculation: shows that incoherent LGS completely fails to provide the propagative beacon needed for compensation.
- Diffraction to the rescue: diffraction restores some coherency via propagation. This is expressed in the van Cittert-Zernike theorem.
- Statement of the van-Cittert-Zernike theorem. Example: Michelson stellar interferometer.
- Application to LGS: the bigger the beacon (diameter at altitude) the worse the effect of incoherency. The smaller the beacon, the worse the effect of focal anisoplanatism.
- Multiple LGS (a constellation) is equivalent to increasing the incoherent beacon diameter.
- Suggestion: Use slightly different colors for each Rayleigh LGS component to distinguish components. Process the Sodium LGS constellation components sequentially to distinguish components.
- Further consequences of Van Cittert-Zernike: Same effect that wipes out incoherency at LGS backscatter altitude can also wipe out effects of high altitude turbulence (scidar results). Incoming plane waves from astronomical objects equivalent to big beacons. Difference between LGS and astro object plane wave (i.e. focal anisoplanatism) means high altitude turbulence has different effects on astro source and LGS, and therefore is not corrected completely.